

# Analytical Solutions to the Mass-Anisotropy Degeneracy with Higher Order Jeans Analysis: A General Method

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## ABSTRACT

The Jeans analysis is often used to infer the total density of a system by relating the velocity moments of an observable tracer population to the underlying gravitational potential. This technique has recently been applied in the search for Dark Matter in objects such as dwarf spheroidal galaxies where the presence of Dark Matter is inferred via stellar velocities. A precise account of the density is needed to constrain the expected gamma ray flux from DM self-annihilation and to distinguish between cold and warm dark matter models. Unfortunately the traditional method of fitting the second order Jeans equation to the tracer dispersion suffers from an unbreakable degeneracy of solutions due to the unknown velocity anisotropy of the projected system. To tackle this degeneracy one can appeal to higher moments of the Jeans equation. By introducing an analog to the Binney anisotropy parameter at fourth order,  $\beta'$  we create a framework that encompasses all solutions to the fourth order Jeans equations. The condition  $\beta' = f(\beta)$  ensures that the degeneracy is lifted and we interpret the separable augmented density system as the order-independent case  $\beta' = \beta$ . For a generic choice of  $\beta'$  we present the line of sight projection of the fourth moment and how it could be incorporated into a joint likelihood analysis of the dispersion and kurtosis. The framework is then extended to all orders such that constraints may be placed to ensure a physically positive distribution function. Having presented the mathematical framework, we then use it to make preliminary analyses of existing data leading to interesting results which strongly motivate further study.

**Key words:** galaxies: kinematics and dynamics– dwarf–Local Group –cosmology: dark matter

## 1 INTRODUCTION

The favoured  $\Lambda$ CDM model of cosmology is consistent with a large invisible non-baryonic component of matter. To infer its existence astronomers thus look for the gravitational effect of its significant mass upon luminous tracer objects or for the observable products of DM annihilation and/or decay such as gamma rays (Gunn et al. 1978; Stecker 1978) which have been used in recent searches (e.g Abdo et al. 2010) for dark matter. In both instances the density distribution of the system is critical with the Earth-incident flux of annihilation products dependent not only on model-dependent properties derived from particle physics (see e.g. Pieri et al. 2009) but also on the square of the density distribution of dark matter within the source. This is encoded in what is known as the *astrophysical J-factor* which can be written

(Walker et al. 2011),

$$J(\theta_{\text{int}}) = \frac{4\pi}{d^2} \int_0^{\theta_{\text{int}}} r^2 \rho_{\text{dm}}^2(r) dr \quad (1)$$

where  $d$  is the distance to the source,  $\rho_{\text{dm}}(r)$  is the local density of dark matter and  $\theta_{\text{int}}$  is the integration angle which is related to a given solid angle of the source via  $\Delta\Omega = 2\pi(1 - \cos \theta_{\text{int}})$ . The quadratic dependence upon the density introduces a very significant and DM model independent contribution to the flux which makes the choice of astrophysical source critically (Walker & Penarrubia 2011) important for optimising DM searches. Though one might expect that the galactic center would provide the strongest signal, the strong and chaotic astrophysical backgrounds make it arguably less favourable than dwarf spheroidal galaxies (dSphs) of the local group which have been identified as having a large mass-to-light discrepancy (Aaronson 1983) suitable for dark matter searches (Lake 1990; Evans et al. 2004). This, in conjunction with their relative proximity to earth, makes them natural laboratories for DM and in recent

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years it has been possible to obtain samples of stellar positions and velocities (e.g Walker et al. 2009a) that are large enough for statistical treatment. Since the typical angular resolution of gamma ray telescopes is larger than the angular size on the sky of dwarf spheroidal galaxies (Abdo et al. 2010), it turns out that the J-factor is not extremely sensitive to the distribution of dark matter in the core of dwarf spheroidals, although in the event of a signal being observed we would ideally want a better indication of the J-factors than the range of current estimates which vary over about an order of magnitude.

Another reason why it is important to study the centre of dwarf spheroidals is to probe another aspect of dark matter, namely its primordial velocity. In the cold dark matter hypothesis the kinetic energy of dark matter at the start of structure formation is some very small fraction of its rest mass energy (Hofmann et al. 2001; Green et al. 2004) and the smallest structures above this free streaming scale are the first to form. In models of hot dark matter (see e.g. Doroshkevich et al. 1980) dark matter begins completely relativistic and the largest structures form first. A (rather finely tuned) compromise between these two extremes is the idea of warm dark matter (Bond et al. 1982) where dark matter is not created with highly relativistic velocities, but nevertheless with significant velocities, meaning that the normal growth pattern of cold dark matter proceeds only above a length scale related to the free streaming length corresponding to the initial velocity. This idea has been invoked to explain the lack of predicted satellites of the Milky Way (Moore et al. 1999) as well as some interpretations of tracer populations in dwarf spheroidals wherein it is argued that dark matter halos possess a significant core, possibly due to some inherent initial kinetic energy (Gilmore et al. 2007). At the same time, the importance of the role of baryons upon dark matter density in the core of halos is becoming increasingly clear (Governato et al. 2012). Whatever the underlying physics, it is clear that we would like to be able to interpret the stellar velocity dispersion in such objects more effectively.

To infer the DM density from the kinematic data the Jeans analysis is used to relate the joint distribution of tracer stars' positions and velocities to the underlying potential of the system. Traditionally the second order Jeans equation (Binney & Tremaine 2008) is used to generate the velocity dispersions for a set of input parameters including the potential which is then fitted to the dSph data with a likelihood analysis by radially binning the line of sight velocities for the variance and assuming Gaussianity. It has long been known however (Dejonghe 1987; Merritt 1987) that this analysis may not be used to uniquely specify the potential for anisotropic systems for which the variances of the radial and tangential velocity components are not equal. As it is only possible to observe the projected quantities along the line of sight, the intrinsic dispersions of the system are convolved such that there is a degeneracy of indistinguishable solutions to the Jeans analysis. Indeed it has been shown that the observed line of sight dispersion can be generated by any given parameterisation of the anisotropy parameter (Evans et al. 2009) thus leaving the potential almost completely unconstrained. This is the so-called Jeans degeneracy problem which is the main subject of this work.

A discussion of the higher order moments is presented

herein with a mathematical description of how they enter the Jeans analysis and what assumptions are required to ensure that the Jeans degeneracy is solved or at least partially lifted. This is then placed into the practical context of a joint likelihood analysis of the variance and kurtosis in dwarf spheroidal galaxies. We evaluate the contribution by Lokas (2002) in establishing a model for the kurtosis that may be used to lift the degeneracy (Lokas et al. 2005) and extend the method to general anisotropy as proposed by An (2011b) with the separable augmented density system.

To simplify the mathematical description of the higher order Jeans equations there has been much success in the literature since the advent of the augmented density formalism by Dejonghe (1986). Whilst an application (Dejonghe 1987; Baes & van Hese 2007) of this method has generally been limited to models (Plummer 1915; Hernquist 1990) with particularly simple potential-density pairs, the recent work of An (2011b) describes for generic density and anisotropy, a separable system that solves the Jeans degeneracy problem completely by specifying moments at all orders with the potential and anisotropy parameter alone. This is however by no means a general solution and without a strong physical motivation its practical use is limited. The layout of our paper is as follows. In section 2 we review the mathematics behind the Jeans equation and line of sight calculations. To utilise the constraining power of the fourth order statistics we are thus motivated to provide the full analytical set of fourth order solutions which is achieved with the introduction of an analog to the Binney anisotropy parameter at fourth order. This is outlined in section 3 wherein we show how to construct a generic model for the projected fourth order moment and how one could incorporate it into a joint likelihood analysis, extending to full generality the over-constrained method employed by Lokas et al. (2005). In section 4 we use data from the Fornax galaxy to construct probability distributions for both the J-factor and the density of dark matter at 100pc from the galactic centre assuming an NFW profile. We contrast directly the results of the traditional and joint analysis and by direct comparison of fourth order models are able to evaluate the robustness of observables to the choice of  $\beta'$ . Finally we will make some concluding remarks and outline our future research program.

## 2 PRELIMINARY

### 2.1 Moments of the Distribution Function

In the study of stellar systems, a 6-dimensional function  $f$  is used to specify the distribution (Jeans 1915) of stars in position and velocity space. For a spherically symmetric system this is related to the underlying gravitational potential  $\Phi(r)$  by the time-independent and collisionless Boltzmann equation (Merrifield & Kent 1990),

$$\begin{aligned}
 \frac{\partial f}{\partial t} &= v_r \frac{\partial f}{\partial r} + \left( \frac{v_\theta^2 + v_\phi^2}{r} - \frac{d\Phi}{dr} \right) \frac{\partial f}{\partial v_r} \\
 &+ \frac{1}{r} (v_\phi^2 \cot \theta - v_r v_\theta) \frac{\partial f}{\partial v_\theta} \\
 &- \frac{1}{r} (v_\phi v_r + v_\phi v_\theta \cot \theta) \frac{\partial f}{\partial v_\phi} \\
 &= 0.
 \end{aligned} \tag{2}$$

Multiplying (2) by  $v_r^l v_\theta^m v_\phi^n$  and then integrating over all velocities restates it in terms of its *true* velocity moments

$$\overline{\nu v_r^{2i} v_\theta^{2j} v_\phi^{2k}} = \int v_r^{2i} v_\theta^{2j} v_\phi^{2k} f(r, \mathbf{v}) d^3 v. \tag{3}$$

where  $\nu(r)$ , as an effective zeroth moment that marginalises the distribution function in velocity space, is the local density of stars. Due to the spherical symmetry of the system it is trivial to show by averaging over the azimuthal angles that the odd moments vanish and that many of the true even moments are related by constant prefactors. To make the notation more compact we again follow the example of Merrifield & Kent (1990) and introduce the *intrinsic* moments of the tangential velocity  $v_t = (v_\theta^2 + v_\phi^2)^{1/2}$ ,

$$\overline{v_r^{2i} v_\theta^{2j} v_\phi^{2k}} = \frac{1}{\pi} B(j + \frac{1}{2}, k + \frac{1}{2}) \overline{v_r^{2i} v_t^{2(j+k)}} \tag{4}$$

where  $B(x, y)$  is the Beta function. In the subsequent analysis we find that it is not necessary to explicitly refer to the true moments at any stage and to simplify the mathematics the tangential moments will be used exclusively from here on in.

## 2.2 Jeans Equations

To isolate the second order moments i.e the radial and tangential dispersions which in practice have the smallest statistical errors, the Boltzmann equation is traditionally multiplied by  $v_r$  and integrated over all velocities (Binney & Tremaine 2008) to give,

$$\frac{d(\nu \sigma_r^2)}{dr} + \frac{2\beta}{r} \nu \sigma_r^2 + \nu \frac{d\Phi}{dr} = 0. \tag{5}$$

where  $\nu(r)$  is the local stellar density,  $\Phi(r)$  is the gravitational potential that depends on the total density of the system  $\rho(r) = \nu(r) + \rho_{\text{dm}}(r)$  via,

$$\Phi(r) = \frac{4\pi G}{r} \int_0^r r'^2 \rho(r') dr' \tag{6}$$

and the Binney anisotropy parameter (Binney & Tremaine 2008)  $\beta(r)$ ,

$$\beta(r) \equiv 1 - \frac{\sigma_t^2(r)}{2\sigma_r^2(r)}, \tag{7}$$

measures the deviation of the dispersions from the isotropic system ( $\sigma_r^2 = \sigma_\theta^2 = \frac{1}{2}\sigma_t^2$ ) wherein all directions in velocity space are equally probable<sup>1</sup>. For a dSph, where the mass-luminosity ratios are often greater than 10 (Mateo 1998) the dark matter component is very significant.

<sup>1</sup> We choose for mathematical convenience to adopt the 2D tangential dispersion rather than the 1D employed by e.g. Lokas (2002) which accounts for the additional factor of 2.

To illustrate the higher order analysis we consider first the fourth order where multiplying equation (2) by  $v_r^3$  and  $v_r v_\theta^2$  respectively relates the three intrinsic moments at fourth order  $\overline{v_r^4}$ ,  $\overline{v_t^4}$  and  $\overline{v_r^2 v_t^2}$  by the two fourth order Jeans equations (Merrifield & Kent 1990),

$$\frac{d(\overline{\nu v_r^4})}{dr} - \frac{3}{r} \overline{\nu v_r^2 v_t^2} + \frac{2}{r} \overline{\nu v_r^4} + 3\nu \sigma_r^2 \frac{d\Phi}{dr} = 0 \tag{8}$$

$$\frac{d(\overline{\nu v_r^2 v_t^2})}{dr} - \frac{1}{r} \overline{\nu v_t^4} + \frac{4}{r} \overline{\nu v_r^2 v_t^2} + \nu \sigma_t^2 \frac{d\Phi}{dr} = 0. \tag{9}$$

The advent of the augmented density system by Dejonghe (1986) has greatly enhanced the mathematical description of the Jeans analysis and it is within this framework that the complete set of Jeans equations has been presented (An 2011a),

$$\begin{aligned}
 \frac{d(\overline{\nu v_r^{2p} v_t^{2q}})}{dr} &= -\frac{2}{r} \left[ (q+1) \overline{\nu v_r^{2p} v_t^{2q}} - (p - \frac{1}{2}) \overline{\nu v_r^{2p-2} v_t^{2q+2}} \right] \\
 &- (2p-1) \overline{\nu v_r^{2p-2} v_t^{2q}} \frac{d\Phi}{dr}.
 \end{aligned} \tag{10}$$

The number of equations at  $2n$ th order is therefore  $n$ , the number of permutations of  $(p, q)$  for which  $p + q = n$  and  $1 \leq p \leq n$ ,  $0 \leq q \leq n$ . Each of the  $n$  moments at  $2n$ th order enter the derivative of a corresponding equation apart from  $v_t^{2n}$ .

## 2.3 Projected Moments

Unfortunately due to the distant nature of astronomical objects the stellar positions and velocities may only be observed along the line-of sight and rather than the true moments and the local density one must instead use the *projected* moments and the surface density profile  $\Sigma$  as functions of the projected radius  $R$ . The surface density profile is the projection along the line of sight of  $\nu(r)$ ,

$$\Sigma(R) = 2 \int_R^\infty \frac{\nu(r) r}{\sqrt{r^2 - R^2}} dr. \tag{11}$$

which may be inverted directly via the Abel Inversion for the local density. The component of the velocity along the line of sight, which for convenience is chosen as the z-direction, may be expressed as

$$v_{\text{los}} = v_r \cos(a) - v_\theta \sin(a) = v_r \sqrt{1 - \frac{R^2}{r^2}} - v_\theta \frac{R}{r}. \tag{12}$$

where  $\sin a = R/r$  is the unobservable depth angle that determines the extent to which each stellar velocity is radial or tangential. Finding the projected moment at  $2n$ th order (see also Dejonghe & Merritt 1992) is akin to averaging over the  $2n$ th power of equation (12),

$$\begin{aligned}
 \Sigma \overline{v_{\text{los}}^{2n}}(R) &= 2 \int_R^\infty \overline{(v_r \cos a - v_\theta \sin a)^{2n}} \frac{\nu(r) r}{\sqrt{r^2 - R^2}} dr \\
 &= 2 \int_R^\infty \sum_{k=0}^n C_{n,k} \overline{v_r^{2(n-k)} v_t^{2k}} \frac{\nu(r) r}{\sqrt{r^2 - R^2}} dr
 \end{aligned} \tag{13}$$

and one thus requires all of the intrinsic moments to calculate the projection with the coefficients,

$$C_{n,k} = \binom{2n}{2k} \frac{B(k + \frac{1}{2}, \frac{1}{2})}{\pi} \left( 1 - \frac{R^2}{r^2} \right)^{n-k} \left( \frac{R^2}{r^2} \right)^k. \tag{14}$$

To simplify the projected dispersion the anisotropy parameter is again introduced in place of the tangential dispersion,

$$\Sigma\sigma_{\text{los}}^2(R) = 2 \int_R^\infty (1 - \beta \frac{R^2}{r^2}) \frac{\nu\sigma_r^2 r}{\sqrt{r^2 - R^2}} dr \quad (15)$$

such that specifying the potential and anisotropy parameter is sufficient to first calculate the radial dispersion via equation (5) and then the projected moment for comparison with observation. An explicit calculation of the projected fourth moment

$$\Sigma\overline{v_{\text{los}}^4}(R) = 2 \int_R^\infty \left( C_{2,0}\overline{v_r^4} + C_{2,1}\overline{v_r^2 v_t^2} + C_{2,2}\overline{v_t^4} \right) \frac{\nu(r)r}{\sqrt{r^2 - R^2}} dr \quad (16)$$

$$C_{2,k} = \begin{cases} (1 - \frac{R^2}{r^2})^2 & k = 0 \\ 3 \frac{R^2}{r^4} (r^2 - R^2) & k = 1 \\ \frac{3}{8} \frac{R^4}{r^4} & k = 2 \end{cases} \quad (17)$$

recovers the result in Merrifield & Kent (1990).

## 2.4 The Degeneracy Problem

The normal Jeans analysis, which approximates the distribution function by its second order moments, has traditionally been the only viable means of modeling the limited sample sizes afforded by kinematic surveys of dwarf spheroidal galaxies. Unfortunately it has been demonstrated (e.g. Merritt 1987) that the integral equation (15) can be highly degenerate with no way of distinguishing the entangled intrinsic dispersions (although see (Walker & Penarrubia 2011) for an interesting recent attempt to break this degeneracy). In a typical statistical treatment, where a model for the dark matter density is fitted with a hand-picked form of the anisotropy, there are numerate parameter sets  $p = \{\beta(r), \Phi(r)\}$  that yield identical line of sight dispersions within statistical errors. As the anisotropy parameter is a completely unknown degree of freedom it is impossible to uniquely specify the potential with the line-of-sight dispersion alone and such a treatment is liable not only to imprecision but also to inaccuracy. With a recent improvement in the data, there has been a renewed interest in the higher order moments that may be used to distinguish those parameter sets degenerate at second order. Unfortunately whilst the fourth moment is practically within reach, a theoretical problem persists. As suggested by An (2011a) we note that at each successive order the Jeans analysis introduces  $n + 1$  moments and only  $n$  constraining Jeans equations such that the intrinsic moments are not specified by the second order parameters  $p$ . A minimal requirement to define the system at fourth order is to specify one of the fourth order moments or, in analogy with Binney's anisotropy parameter, to specify the ratio of two. To lift the degeneracy one additionally desires that the projected fourth moment depend only upon the second order parameters such that a new net constraint is added to the system without expanding the parameter space. The anisotropy parameter in particular, whilst inherent to the dispersions, has no direct bearing on the higher order moments without artificial insertion. It can be concluded therefore that to utilise the higher order moments

one must present a new constraint to the system via a simplifying assumption or empirical observation that optimally ties the higher order moments to the anisotropy parameter and thus may be used to constrain it. To lift the degeneracy completely one requires all of the projected moments which as proved by Dejonghe & Merritt (1992) is equivalent to knowledge of the distribution function.

## 3 ANALYTICAL SOLUTIONS TO THE JEANS DEGENERACY

In this section we discuss analytical models that place additional constraints on the intrinsic fourth order moments for a practical use in tackling the mass-anisotropy degeneracy. One prominent example in the literature is the work of Lokas (2002) who assumes a form (Henon 1973) of the distribution function  $f(E, L) = f_0(E)L^{-2\beta}$  with a constant velocity anisotropy that is separable when expressed in terms of the specific binding energy  $E = -\Phi - \frac{1}{2}(v_r^2 + v_t^2)$  and the specific angular momentum  $|L| = rv_t$ . With this assumption then without specifying the form of the energy component  $f_0(E)$  it is possible via equation (3) to calculate the ratio of higher order moments with the anisotropy parameter. The projected fourth moment is thus shown to be dependent only on the second order parameters and is applied to the degeneracy problem for the Draco galaxy (Lokas et al. 2005). It has been argued however (An 2011a) that this model, whilst relatively clear to interpret, over-constrains the problem and may be extended to general anisotropy with the separable augmented density model. As such we outline this model and provide a formula for the projected moments. This system however only provides one solution and whilst mathematically elegant it does not yet have a strong physical motivation. We therefore provide a framework without appealing to the augmented density formalism directly, that encompasses all solutions to the fourth order by introducing an analog to the anisotropy parameter at fourth order. The aim of this framework is both to evaluate the particularly convenient separable augmented solution and to facilitate more physically motivated models perhaps inspired by empirical observation. The framework is then extended to all orders.

### 3.1 The Separable Augmented Density Model

A full account of the augmented density formalism is not necessary for the wider context of this paper but we direct the interested reader to the literature (Dejonghe 1986; Dejonghe & Merritt 1992; An 2011a,b). Here we give a brief outline and then present the main findings relevant to a study of dwarf spheroidal galaxies in the simplest possible terms. By careful consideration of equation (3) and the Jeans theorem (Jeans 1915) it was noted by Dejonghe (1986) that the degeneracy of distribution functions which fit the local density  $\nu(r)$  can be represented by *augmenting* the density into an infinite set of degenerate bi-variate functions of the *augmented configuration space*  $(\Psi, r)$  and that providing a specific form of the augmented density is equivalent to providing the distribution function. Here  $\psi$  is the positive binding specific potential defined by  $\frac{d\Psi}{dr} = -\frac{d\Phi}{dr}$ .

Moments are also augmented and to retrieve the observables the augmented quantities are *deaugmented* by taking the limit  $\Psi \rightarrow \Psi(r)$ . The power of the method lies in the ease with which one may relate the augmented quantities and in augmented configuration space the Jeans equations are replaced by the simple partial differential equations (Dejonghe & Merritt 1992),

$$\frac{\partial}{\partial \Psi} \overline{\nu v_r^{2p} v_t^{2q}}(\Psi, r) = (2p-1) \overline{\nu v_r^{2p-2} v_t^{2q}}(\Psi, r) \quad (18)$$

$$\frac{\partial}{\partial r^2} \left[ r^{2q+2} \overline{\nu v_r^{2p} v_t^{2q}}(\Psi, r) \right] = (p - \frac{1}{2}) r^{2q} \overline{\nu v_r^{2p-2} v_t^{2q+2}}(\Psi, r) \quad (19)$$

and the augmented total derivative An (2011a),

$$\frac{d}{dr} \rightarrow \frac{\partial}{\partial r} + \frac{\partial}{\partial \Psi} \frac{d\Psi}{dr}. \quad (20)$$

To determine the moment ratios at fixed order it is necessary only to consider the order preserving equation (19) such that for a system with a separable augmented density,

$$\tilde{\nu}(\Psi, r) = P(\Psi)R(r) \quad (21)$$

the anisotropy is dependent only on the radial component  $R(r)$  and this has been demonstrated by An (2011a) who shows that

$$\beta = -\frac{d \log R}{d \log r^2}. \quad (22)$$

An even more profound property of the separable system (equation 21) is that the ratio of moments at a given order scales only with the constant prefactor inherent to the isotropic system. In practice this has the notable effect of collapsing all of the Jeans equations to just one (An 2011a),

$$\frac{d(\overline{\nu v_r^{2n}})}{dr} + \frac{2\beta}{r} \overline{\nu v_r^{2n}} + (2n-1) \overline{\nu v_r^{2n-2}} \frac{d\Phi}{dr} = 0 \quad (23)$$

that enables a unique calculation of the radial intrinsic moment and thus absorbs the additional degree of freedom in the Jeans analysis. In calculation of the other moments one may use (An 2011a),

$$\overline{\nu v_r^{2(n-k)} v_t^{2k}} = \frac{\alpha_k}{(n-k+\frac{1}{2})_k} \overline{\nu v_r^{2n}} \quad (24)$$

where  $(a)_k = \prod_{i=1}^k (a+i-1)$  is the Pochhammer symbol that describes the *rising sequential product*,  $\alpha_0 = 1$  and

$$\alpha_{q+1} = (q+1-\beta)\alpha_q + \frac{r}{2} \frac{d\alpha_q}{dr} \quad (25)$$

can be used to iteratively generate all moment ratios noting again that one requires only the anisotropy parameter. The projected moments then follow trivially,

$$\Sigma \overline{\nu v_{los}^{2n}}(R) = 2 \int_R^\infty \sum_{k=0}^n C_{n,k} \frac{\alpha_k}{(n-k+\frac{1}{2})_k} \frac{\overline{\nu v_r^{2n}} r}{\sqrt{r^2 - R^2}} dr \quad (26)$$

and we thus demonstrate that the fourth projected moment depends only on the second order parameters. In practice one must first solve  $n$  differential equations to calculate in turn each radial intrinsic moment via the Jeans equation (23) and then  $n$  iterations of (25) for a theoretical prediction of the projected moment. As an example consider the projected fourth moment for which we require,

$$\alpha_1 = 1 - \beta, \quad \alpha_2 = (1 - \beta)(2 - \beta) - \frac{r}{2} \frac{d\beta}{dr}. \quad (27)$$

and the fourth order Jeans equation

$$\frac{d\nu \overline{v_t^4}}{dr} + \frac{2\beta}{r} \overline{\nu v_r^4} + 3\nu \sigma_r^2 \frac{d\phi}{dr} = 0. \quad (28)$$

Evaluating (26) with  $n = 2$  yields after some algebra,

$$\Sigma \overline{v_{los}^4}(R) = 2 \int_R^\infty g(\beta, r, R) \frac{\overline{\nu v_r^4} r}{\sqrt{r^2 - R^2}} dr \quad (29)$$

$$g(\beta, r, R) = 1 - 2\beta \frac{R^2}{r^2} + \frac{\beta(1-\beta)}{2} \frac{R^4}{r^4} - \frac{R^4}{4r^3} \frac{d\beta}{dr} \quad (30)$$

which generalises the result of Lokas (2002) and may be used to tackle the Jeans degeneracy for an arbitrary radial dependence of the anisotropy parameter via the method employed in Lokas et al. (2005). In practice then one must first find the radial dispersion via the second order Jeans equation (5), solve equation (28) for the fourth radial moment and then with the recursive relations (27) it is possible to evaluate the projected moment with equation (16).

In summary the separable augmented density system completely lifts the Jeans degeneracy by providing an infinite set of projected moment equations that one can, in principle, calculate from the potential and the anisotropy at second order. The origin of the system is however purely mathematical and whilst particularly convenient to use it is not unique in exhibiting this behaviour. Indeed one could artificially introduce any arbitrary relationship between the anisotropy parameter and the higher order moments to the same effect. This model, which generates only one DF for a given set of anisotropy and density parameters, is thus a severe restriction to apply to the system without physical motivation. The aim then is to provide a framework that explores the full range of solutions to the Jeans degeneracy problem without restriction from mathematical considerations.

### 3.2 An Extended Model of Anisotropy

In this subsection we will demonstrate, without appealing explicitly to the augmented density formalism, that a unique set of all deprojected moments is specified by introducing an analog to the Binney anisotropy parameter at each successive order. Indeed one can represent all possible distribution functions in this way and we will show that a distribution function can be defined as an infinite set of anisotropy parameter analogs  $\{\beta_n\}$ ,  $1 \leq n \leq \infty$ . If each of these is assumed to have a known relation to the anisotropy parameter then the Jeans degeneracy is completely lifted. For a particular definition of anisotropy parameters, inspired by the separable augmented density system, we provide formulae for the full set of intrinsic moments and their subsequent projected moments. We are then able to interpret the separable augmented density system as a particular subset of this system for which the anisotropy parameter analogs are independent of order  $\{\beta_n\} = \beta \forall n$ .

#### 3.2.1 Fourth Order

To measure the anisotropy at fourth order we choose for simplicity the adjacent moment ratio to the radial fourth

moment and introduce,

$$\beta'(r) = 1 - \frac{3}{2} \frac{\overline{v_r^2 v_t^2}}{\overline{v_r^4}}, \quad (31)$$

which is an analog of the Binney anisotropy parameter for the fourth order that measures the deviation from the isotropic system (A8) where  $\overline{v_r^4} = \frac{3}{2} \overline{v_r^2 v_t^2}$ . Substituting this parameter into the Jeans equation (8),

$$\frac{d(\nu \overline{v_r^4})}{dr} + \frac{2\beta'}{r} \nu \overline{v_r^4} + 3\nu \sigma_r^2 \frac{d\Phi}{dr} = 0, \quad (32)$$

which then, given a functional form for  $\beta'$ , enables the radial fourth moment to be determined uniquely. The choice of equation (31) is now clear as we note that the separable augmented density system thus corresponds to  $\beta = \beta'$ . The mixed moment then follows trivially from the definition of  $\beta'$  and the tangential moment may be calculated via equation (9) as follows. Firstly equation (31) indicates that,

$$\frac{d(\nu \overline{v_r^2 v_t^2})}{dr} = \frac{2}{3} \left[ (1 - \beta') \frac{d(\nu \overline{v_r^4})}{dr} - \frac{d\beta'}{dr} \nu \overline{v_r^4} \right] \quad (33)$$

which may be simplified further with equation (32) by substituting in the derivative of the radial fourth moment,

$$\frac{d(\nu \overline{v_r^2 v_t^2})}{dr} = \frac{2}{3} (1 - \beta') \left[ -\frac{2\beta'}{r} \nu \overline{v_r^4} - 3\nu \sigma_r^2 \frac{d\Phi}{dr} \right] - \frac{2}{3} \frac{d\beta'}{dr} \nu \overline{v_r^4}. \quad (34)$$

Plugging this into the Jeans equation (9) and rearranging for the fourth tangential moment yields,

$$\overline{v_t^4} = \frac{4}{3} \left( (1 - \beta')(2 - \beta') - \frac{r}{2} \frac{d\beta'}{dr} \right) \overline{v_r^4} + 2(\beta' - \beta) r \sigma_r^2 \frac{d\Phi}{dr} \quad (35)$$

which shows that the system is uniquely determined by the second order parameters plus the anisotropy at fourth order. One also notes that in the limit  $\beta' = \beta$ , the tangential moment recovers the result  $\overline{v_t^4} = \frac{4}{3} \alpha_2 \overline{v_r^4}$  for the separable augmented density system. With all the moments one can then compute the projected moment from equation (16) which after some algebra is,

$$\Sigma \overline{v_{los}^4}(R) = 2 \int_R^\infty \left( g(\beta') \overline{v_r^4} + \frac{3R^4}{4r^3} (\beta' - \beta) \sigma_r^2 \frac{d\Phi}{dr} \right) \frac{\nu(r)r}{\sqrt{r^2 - R^2}} dr \quad (36)$$

where  $g(\beta')$  is adapted from equation (30) with  $\beta \rightarrow \beta'$ . An arbitrary choice of  $\beta'$  in addition to the second order parameters represents the complete set of fourth order systems.

Therefore to lift the Jeans degeneracy one must additionally impose that  $\beta' = f(\beta)$  where a specification of the function  $f(\beta)$  defines the model. This is the fundamental concept of this paper. In section 4 we explore the effect of this choice on the key physical observables of the dark matter problem. If one can use empirical evidence to find a correlation between the anisotropy at both orders or indeed a well motivated physical argument for the specific form that such a relationship would take, then the Jeans degeneracy problem can be at least partially lifted with available data. To lift the degeneracy completely, all projected moments are required and in the following section the framework presented above is extended.

### 3.2.2 Anisotropy of the Entire System

To generalise the method in the previous section we again choose to define the anisotropy parameter analog at  $2n$ th order via the adjacent moment ratio to the radial moment,

$$\beta_n(r) = 1 - (n - \frac{1}{2}) \frac{\overline{v_r^{2n-2} v_t^2}}{\overline{v_r^{2n}}} \quad (37)$$

which measures the departure from the isotropic system,  $\overline{v_r^{2n}} = (n - 1/2) \overline{v_r^{2n-2} v_t^2}$  and we note that Binney's anisotropy parameter  $\beta = \beta_1$  and its fourth order analog  $\beta' = \beta_2$  are naturally incorporated into the analysis. Substituting this into the first Jeans equation at  $2n$ th order,

$$\frac{d(\nu \overline{v_r^{2n}})}{dr} + \frac{2\beta_n}{r} \nu \overline{v_r^{2n}} + (2n - 1) \nu \overline{v_r^{2n-2}} \frac{d\Phi}{dr} = 0, \quad (38)$$

which uniquely determines the radial moment if one assumes knowledge of all moments at preceding order. The definition of the separable augmented density system is then extended to  $\{\beta_n\} = \beta$ ,  $\forall n$  and thus represents the system for which the anisotropy parameters are independent of order. We will show that calculation of the other moments then follows via a recursive passage through the remaining Jeans equations. To generalise the method used to obtain the tangential moment in equation (35) we consider the generic Jeans equation (10) and introduce, where convenient, the compact notation,

$$\overline{\nu v_r^{2p} v_t^{2q}} = m_{p,q}. \quad (39)$$

and the adjacent moment ratios at order  $2n = 2(p + q)$ ,

$$f_q^{p+q}(\{\beta_n\}, r) \equiv (p + \frac{1}{2}) \frac{m_{p,q}}{m_{p+1,q-1}} \quad (40)$$

where we note that by definition  $f_1^n = 1 - \beta_n$ . The prefactor  $p + 1/2$  is included to absorb the order dependence inherent to the isotropic system (A8) such that we can isolate the order dependence of the systems anisotropy. Rearranging equation (38) for the unknown moment,

$$(p - \frac{1}{2}) \overline{\nu v_r^{2p-2} v_t^{2q+2}} = \frac{r}{2} \frac{d(\nu \overline{v_r^{2p} v_t^{2q}})}{dr} + (q + 1) \overline{\nu v_r^{2p} v_t^{2q}} \frac{d\Phi}{dr} \quad (41)$$

such that upon substituting the prior moment ratio (40) into the derivative,

$$\begin{aligned} \frac{dm_{p,q}}{dr} &= \frac{1}{p + \frac{1}{2}} \left( m_{p+1,q-1} \frac{df_q^n}{dr} + f_q^n \frac{dm_{p+1,q-1}}{dr} \right) \\ &= \left[ \frac{1}{f_q^n} \frac{df_q^n}{dr} + \frac{2}{r} (f_q^n - q) \right] m_{p,q} - 2f_q^n m_{p,q-1} \frac{d\Phi}{dr} \end{aligned} \quad (42)$$

we obtain the recursive relation for  $q > 1$ ,

$$f_{q+1}^n \equiv (p - \frac{1}{2}) \frac{m_{p-1,q+1}}{m_{p,q}} \quad (43)$$

$$\begin{aligned} &= 1 + f_q^n + \frac{r}{2} \frac{1}{f_q^n} \frac{df_q^n}{dr} \\ &+ (n - q - \frac{1}{2}) \frac{m_{p-1,q}}{m_{p,q}} r \frac{d\Phi}{dr} \left[ 1 - \frac{f_q^n}{f_{q-1}^n} \right]. \end{aligned} \quad (44)$$

Starting with  $f_1^n = 1 - \beta_n$  this relation may then be used to iteratively calculate the complete set of intrinsic moments

at  $2n$ th order. The first iteration yields,

$$f_2^n = 2 - \beta_n - \frac{r}{2(1 - \beta_n)} \frac{d\beta_n}{dr} + (n - \frac{3}{2}) \frac{m_{n-q-1,q}}{m_{n-q,q}} r \frac{d\Phi}{dr} \frac{\beta_n - \beta_{n-1}}{1 - \beta_n} \quad (45)$$

which we may use to check that with  $f_1^2 = 1 - \beta'$  and  $f_1^1 = 1 - \beta$ , the tangential fourth moment in equation (35) is recovered via  $\bar{v}_t^4 = \frac{4}{3} f_1^2 f_2^2 \bar{v}_r^4$ . To calculate the projected moments we first note that

$$\frac{m_{n-k,k}}{m_{n,0}} = \prod_{j=1}^q \frac{m_{n-j,j}}{m_{n-j+1,j-1}} = \frac{\prod_{j=1}^q f_j^n}{(p + \frac{1}{2})_q} \quad (46)$$

and that in analogy to equation (25) in the separable augmented density system we may define,

$$\alpha_q^n \equiv \prod_{k=1}^q f_k^n, \quad (47)$$

which satisfies the recursive relation

$$\alpha_{q+1}^n = \alpha_q^n f_{q+1}^n. \quad (48)$$

To check that the system converges to the separable augmented density in the limit  $\{\beta_n\} = \beta$  where the order dependence is removed,

$$f_{q+1}^n(\{\beta_n\}) \rightarrow f_{q+1}^n(\beta) = 1 + f_q + \frac{r}{2} \frac{1}{f_q} \frac{df_q}{dr}. \quad (49)$$

we substitute equation (49) into equation (48) with  $\alpha_q = f_q \alpha_{q-1}$  to eliminate  $f_q$  such that after some algebra,

$$\begin{aligned} \alpha_{q+1} &= \left( 1 + \frac{\alpha_q}{\alpha_{q-1}} + \frac{r}{2} \frac{\alpha_{q-1}}{\alpha_q} \left[ \frac{1}{\alpha_{q-1}} \frac{d\alpha_q}{dr} - \frac{\alpha_q}{\alpha_{q-1}^2} \frac{d\alpha_{q-1}}{dr} \right] \right) \alpha_q \\ &= \left( 1 + \frac{\alpha_q}{\alpha_{q-1}} - \frac{r}{2} \frac{1}{\alpha_{q-1}} \frac{d\alpha_{q-1}}{dr} \right) \alpha_q + \frac{r}{2} \frac{d\alpha_q}{dr}, \end{aligned} \quad (50)$$

which one can prove inductively is satisfied by

$$\alpha_{q+1} = (q + k) \alpha_q + \frac{r}{2} \frac{d\alpha_q}{dr}. \quad (51)$$

Applying the boundary condition  $\alpha_1 = 1 - \beta$  and choosing  $\alpha_0 = 1$  yields  $k = 1 - \beta$  and therefore recovers equation (25). To generalise the projected moments in equation (26) we simply promote the ratios to  $\alpha_k^n$ . Whilst seemingly a subtle change, this makes the subsequent calculation considerably more cumbersome. Though the generalisation at fourth order is straightforward the higher order projected moments become rather inconvenient for practical use when one deviates from the separable augmented density system. The inclusion of the higher order analysis is however useful to ensure that the set  $\{\beta_n\}$  yields positive moments at all orders and thus a physical distribution function. This constraint on the system may be represented simply as,

$$f_q^n \geq 0, \quad \forall n, q, \quad (52)$$

which imposes physical constraints on the analogs including  $\beta_n \leq 1$  for  $q = 1$  and for example the *sufficient* but not strictly necessary condition for constant anisotropy  $\beta_n \geq \beta_{n-1}$  corresponding to  $q = 2$ .

### 3.3 Summary

By introducing an analog of the Binney anisotropy parameter at fourth order it is possible to represent the complete

analytic set of projected fourth moments. Not only is the result presented in Lokas (2002) adapted in equation (36) to an arbitrary specification of the anisotropy parameter but as a subset of the separable augmented density system it is interpreted as the particular case where anisotropy is order-independent. To employ the method in Lokas et al. (2005) to lift the Jeans degeneracy one may choose any arbitrary specification  $\beta' = f(\beta)$  to construct a physical model thus removing the constraints upon the distribution function imposed by the separable augmented density. Additionally the higher order projected moments are presented such that constraints (52) analogous to those presented in An (2011a) for the separable augmented density system,  $\alpha_q \geq 0$ , may be used to ensure that any given model has a physical DF. Though the maths underlying the higher order moments quickly becomes impractical as one deviates from the separable augmented density system, the fourth order generalisation remains tractable. With a generic framework we facilitate a model with a stronger physical motivation and state, in general terms, a condition to lift the degeneracy namely a correlation between the anisotropy parameter analogs. Without a strong physical argument to provide this relation analytically we turn to empirical evidence. The additional freedom afforded by the anisotropy parameter analogs provides the means to test models presented in the literature.

## 4 A PRACTICAL APPLICATION TO THE DWARF SPHEROIDAL GALAXY FORNAX

With the extended anisotropy framework a natural progression is to devise a model for  $\beta'$  with a stronger physical basis than the separable augmented density system. Due to the completely unknown nature of the anisotropy this is however a formidable task and at time of writing no such candidate has been found. To implement the above formalism then, we look to dSph data to provide information about higher order anisotropy and as a means of evaluating the models proposed by Lokas (2002) and An (2011b). Recently the kinematic surveys of dSphs in the Local Group have expanded the number of velocity measurements to the order of  $10^3$  which reduces the fourth order statistical errors to tolerable levels.

### 4.1 Data

In our analysis we use data from the Magellan survey (Walker et al. 2009a) and in particular the heliocentric line-of-sight velocities and RA-DEC coordinates for the Fornax galaxy. To remove potential interlopers we use the spectral metallicity data provided in Walker et al. (2009b) and remove all stars with a membership probability of 95 per cent or less retaining a sample of 992 stars. Firstly the Fornax positional data were converted to a projected radius from the centre of the galaxy assuming a distance of  $d = 138$  kpc. This was then used to fit the the local density  $\nu(r)$  to a Plummer profile by matching the surface profile  $\Sigma(R)$  integrated by the area element  $dA = 2\pi R dR$  to the number of stars at radius  $R$  with the result  $r_p = 600$  pc. The velocity

data was then sorted into 12 radial bins of equal size such that there are 12 variance and kurtosis<sup>2</sup> measurements.

#### 4.2 Likelihood Analysis

Though less favoured than the Einasto (Einasto & Haud 1989) and the more general Zhao density profile (Zhao 1996) which generalises the slope parameters of the galaxies central cusp, for simplicity at this early stage of our work we employ the two parameter Navarro-Frenk-White (NFW) profile (Navarro et al. 1996),

$$\rho_{\text{nfw}} = \frac{\rho_0}{\frac{r}{r_n} \left(1 + \frac{r}{r_n}\right)^2} \quad (53)$$

to parameterise the density of dark matter which is sufficient to illustrate the statement of concept presented in this paper. To model the anisotropy parameter we use (Baes & van Hese 2007),

$$\beta(r) = (\beta_\infty - \beta_0) \frac{r^2}{r_\beta^2 + r^2} + \beta_0 \quad (54)$$

which generically describes anisotropy with a quadratic transition about  $r_\beta$  that asymptotes at  $\beta_0$  for  $r = 0$  and  $\beta_\infty$  for  $r \rightarrow \infty$ . In addition to the NFW density parameters  $\{\rho_0, r_n\}$  the content of the full parameter set depends on the nature of the functional form of the anisotropy considered at both orders. We consider five permutations (i) the traditional second order analysis with a single constant anisotropy parameter such that the parameters to be varied are  $p = \{\beta, \rho_0, r_n\}$ , (ii) a second order analysis with a varying anisotropy given by equation (54)  $p = \{\beta_0, \beta_\infty, r_\beta, \rho_0, r_n\}$ , (iii) a fourth order analysis with parameters equivalent to case (i) for which both anisotropy parameters are constant and equal (Lokas 2002) (iv) a fourth order analysis with parameters equivalent to (ii) where both anisotropy parameters have the generalised profile but are equal, i.e the separable augmented density and (v) a fourth order analysis for which both anisotropy parameters are constant but not necessarily equal  $p = \{\beta, \beta', \rho_0, r_n\}$ .

To fit the anisotropy and density parameters  $p$  to the empirical data  $d$  we employ a Monte-Carlo Markov Chain (MCMC) to generate the posterior distribution function from the likelihood,

$$-\ln \mathcal{L}(d|p) = \frac{1}{2} \left\{ \sum_{i=1}^N \frac{(\sigma_{\text{los}}^2 - \hat{\sigma}_{\text{los}}^2)^2}{\alpha_{\sigma^2}^2} + \sum_{i=1}^N \frac{(\kappa_{\text{los}} - \hat{\kappa}_{\text{los}})^2}{\alpha_\kappa^2} \right\} \quad (55)$$

where  $N$  is the number of radial bins,  $X = \{\sigma_{\text{los}}^2, \kappa_{\text{los}}\}$  are the theoretical predictions generated by the parameter set  $p$  for those quantities evaluated at bin radius  $R_i$ ,  $\hat{X}$  is the empirical measurement of  $X$  for the  $i$ th bin of data set  $d$  and  $\alpha_X$  is the error associated with  $X$ . We approximate the unbiased sampling errors by,

$$\alpha_{\sigma^2} = \sqrt{\frac{2}{N-1}} \sigma^2, \quad \alpha_\kappa = \sqrt{\frac{24N(N-2)(N-3)}{(N+1)^2(N+3)(N+5)}} \quad (56)$$

<sup>2</sup> The kurtosis is the normalised fourth moment and we note that the kurtosis of the normal distribution is 3

which for a given bin size are generated by considering the number of times one must sample from a Gaussian distribution to accurately determine its moments. To implement the MCMC the Metropolis-Hastings (Metropolis et al. 1953) algorithm is used to iteratively sample the parameters or simple functions of the parameters for which a uniform sampling is desired. As an example consider the parameter  $\beta$  for which we choose instead  $-\log_{10}[1-\beta]$  to sample uniformly over the range  $[-1, 1]$  in line with the choice for constant anisotropy used in Charbonnier et al. (2011). This choice equivalent to  $-9 \leq \beta \leq 0.9$  ensures that all ratios of the dispersions are uniformly sampled rather than the non-linear phase space of  $\beta$ . With this in mind the following priors and ranges are adopted,

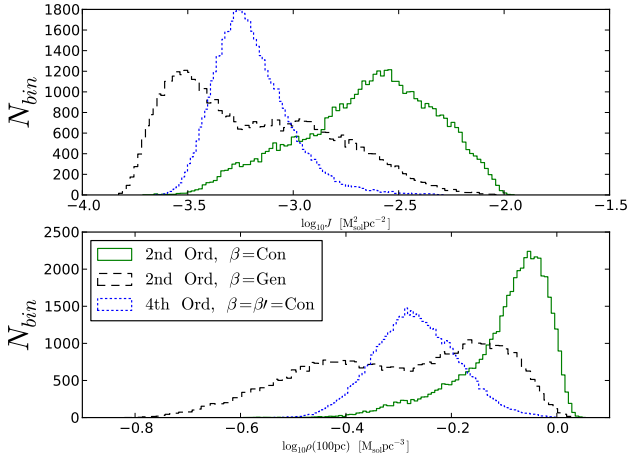
$$\begin{aligned} -\log_{10}[1-A] &: [-1, 1] \\ \log_{10} r_\beta &: [0, 4] \\ \log_{10} \rho_0 &: [-5, 3] \\ \log_{10} r_n &: [1, 5] \end{aligned} \quad (57)$$

where  $A = \{\beta, \beta', \beta_0, \beta_\infty\}$ . To build the MCMC chain a set of parameters  $p'$  is drawn randomly via a Gaussian distribution centered at  $p$ . The set  $p'$  is added to the chain if  $\mathcal{L}(d|p')/\mathcal{L}(d|p) > U$  where  $U$  is generated randomly from the uniform distribution over the range  $[0, 1]$ . The width of the Gaussian known as the *temperature* of each parameter is then adjusted to give an acceptance ratio of approximately 0.2.

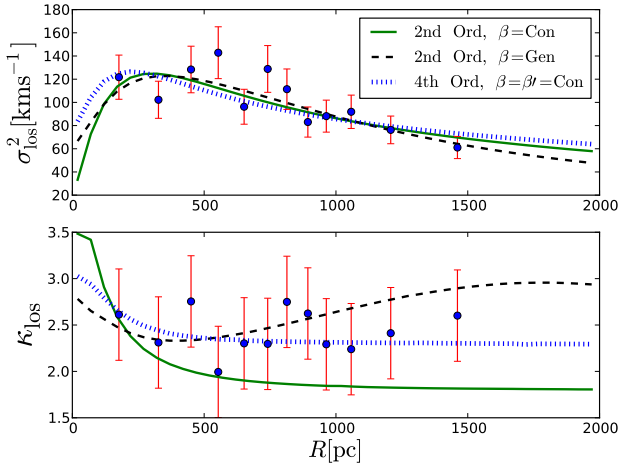
#### 4.3 Results

In analysing the fourth order statistics our aims are twofold. Firstly we intend to demonstrate the constraining power of the fourth order method and by direct comparison with the traditional analysis we also look for inconsistencies. Though this has been achieved in part by the work of Lokas et al. (2005), our second aim, to check the robustness of the fourth order model, is new work. Fig. 1 demonstrates the effect of introducing the fourth moment to the astrophysical J-factor and the DM density profile at 100pc with Lokas's model. Though the width of the distribution is notably reduced for the J-factor the improvement upon the precision of the density is very small. A noticeable discrepancy is observed between the peaks of the J-factor distributions which is relaxed when the second order is extended to general anisotropy. Whilst one might be tempted to hail this as a success of the accuracy introduced by the fourth order statistics, the arbitrary nature with which the model  $\beta = \beta'$  is chosen suggests caution and it is entirely feasible that a different choice could align the two distributions perfectly. Be that as it may, by not fitting to the fourth order at all an assumption is made to the fourth order without reference to the data. In Fig. 2 we demonstrate that if Lokas's model is assumed to be true then the second order fit with equivalently constant anisotropy provides an inaccurate account of the kurtosis which decreases the joint likelihood by a factor of almost 2. A key question then is whether the uncertainty associated with the choice of model at fourth order is significantly less than that which may be alleviated with its inclusion. To check the robustness of the Lokas model we performed a likelihood analysis for two additional models with a simple extension to general anisotropy within the



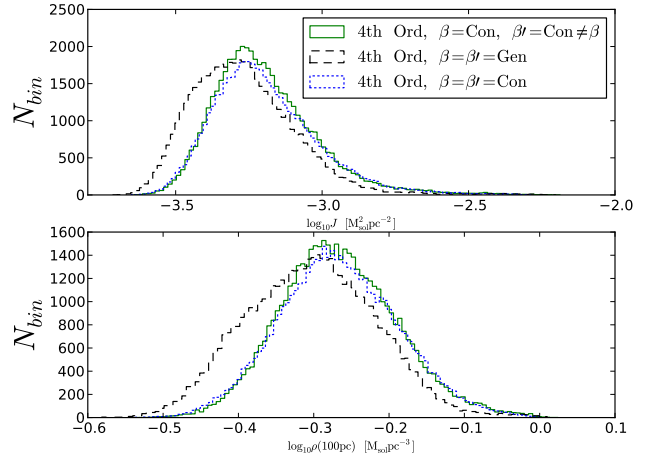


**Figure 1.** Posterior distributions for a point-like approximation to the astrophysical J-factor (1) with an integration angle of  $1^\circ$  (above) and the density of dark matter at 100pc (below) derived from parameters generated via the MCMC for a traditional second order analysis with constant anisotropy (solid), a traditional analysis with generalised anisotropy (54) (dashed) and a joint analysis of dispersions and kurtosis with constant anisotropy (dotted).

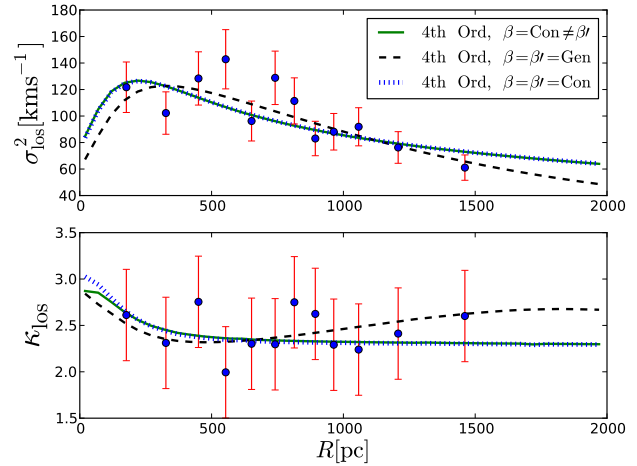


**Figure 2.** Dispersion and Kurtosis profiles for the three models outlined in Fig. 1. The dotted lines show the moments generated by the parameters that maximise the joint likelihood function (55) for 12 equally sized radial bins of the Fornax data with errors defined by equation (56). For the second order models (solid and dashed) the dispersion is generated by parameters that maximise the likelihood for a fit to the dispersions only. The resulting kurtosis profiles are produced with the separable augmented density model of the fourth moment.

separable augmented density framework and a model with constant anisotropy for which we allowed a constant choice to  $\beta'$  to vary freely. The result of this analysis is shown in Figs. 3 and 4 with the surprising result that the additional freedom afforded to the system has little impact on the prediction for the observable quantities shown, indeed drastically less than the variation shown from the second order fit.



**Figure 3.** Distributions as specified in Fig. 1 for joint fits to the dispersion and kurtosis with fourth order models  $\beta'(r) = \beta$  (dotted),  $\beta'(r) = \beta(r)$  (dashed) and for which  $\beta'$  is a constant that is allowed to vary freely (solid).



**Figure 4.** Best fits to the joint likelihood analysis for the three models outlined in Fig. 3 represented with a consistent colour scheme.

Whether this is a property of the data, which being strongly platykurtic excludes large portions of the parameter space, or the model is unclear and warrants further investigation.

## 5 DISCUSSION

The Jeans analysis is extremely useful in identifying the gravitational potential from the line of sight velocities of stars moving in that potential and is used to learn more about many different kinds of astrophysical objects. The information obtained in this way is limited by degeneracies which exist as a result of our ignorance of the velocity anisotropy within the stellar tracer populations. By looking at not only the width (2nd moment of velocity) of the stellar velocities but also their kurtosis (ratio between 2nd

and 4th moment) we are able to break some of these degeneracies to a greater or lesser extent. It has been shown that the additional information provided by the higher order moments can demonstrably tighten the constraints on observables such as the astrophysical J-factor and the core density (Lokas et al. 2005). To date this has only been shown when an arbitrary and quite constraining choice has made to relate the two projected moments to each other.

In this paper we have presented a new mathematical framework for calculating the higher order moments of the Jeans equation based upon introducing an analogue of the Binney Anisotropy parameter at each higher order and we have demonstrated that this completely determines the solution to the Jeans equation at each order. By studying Fornax we demonstrate that a model with constant and order independent anisotropy parameters  $\beta' = \beta$  predicts both the J-factor and the density of dark matter at 100pc from the galactic centre with a smaller uncertainty than when only the second order moments are taken into account. This is essentially a reproduction of the analysis of (Lokas et al. 2005) and reaches the same conclusion as reached by those authors. We then go on to consider two more additional cases which to our knowledge have not been tested against data before. The first is the employment of the separable augmented density, which corresponds to the situation where the second order velocity anisotropy  $\beta(r)$  can be an arbitrary function of radius but where it is equal to its fourth order sibling, i.e.  $\beta'(r) = \beta(r)$ . The second new case we consider is where we allow separate anisotropy parameters for the second and fourth order moments, although for the time being we only consider the case where they are both (different) constants. It appears that the predictions for the J factor and the core density in the three different cases where we have made different assumptions about the second and fourth order velocity anisotropy parameters do not vary a huge amount whichever particular set of assumptions is adopted - simply by including the higher moments of the data, the Jeans analysis seems to become more powerful. Whilst a detailed investigation is required to confirm this finding it provides strong motivation for further study into the higher order Jeans analysis and demonstrates that even a fourth order model without a strong physical basis could prove more constraining than a second order analysis that makes an assumption without reference to the fourth order data.

Ideally we would like to test how constraining the additional fourth order information is in the case where we allow the relationship between  $\beta$  and  $\beta'$  to vary to a greater or lesser extent. It would also be interesting to try and use the results of N-body simulations to motivate physical choices for the relationship between the two anisotropy parameters. Furthermore, it remains to consider different functional forms for the dark matter halo such as the Einasto profile which can describe both cored and cuspy halos in order to see what the higher order information is telling us about the shape of the density profile in the centre of dwarf Spheroidals. We are working on all these issues and hope to present new results in the near future.

## ACKNOWLEDGMENTS

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## APPENDIX A: ORDER DEPENDENCE OF MOMENT RATIOS IN THE ISOTROPIC SYSTEM

Let's assume that the center of the galaxy lies in the positive z-direction for a sphere with corresponding polar angles ( $0 \leq \epsilon_1 \leq \pi$ ,  $0 \leq \epsilon_2 \leq 2\pi$ ) centered at the stellar position such that the radial velocity is treated as  $v_z$  and the angular velocities as  $v_x$  and  $v_y$ . We may thus write,

$$v_\theta = v \sin \epsilon_1 \cos \epsilon_2 \quad (\text{A1})$$

$$v_\phi = v \sin \epsilon_1 \sin \epsilon_2 \quad (\text{A2})$$

$$v_r = v \cos \epsilon_1 \quad (\text{A3})$$

where  $v$  is the total stellar velocity  $v^2 = v_r^2 + v_\theta^2 + v_\phi^2$  and we note that  $v_t = v \sin \epsilon_1$ . The moments may thus be expressed as,

$$\overline{v_r^{2p} v_t^{2q}} = \overline{v^{2(p+q)} \cos^{2p} \epsilon_1 \sin^{2q} \epsilon_1} \quad (\text{A4})$$

such that to calculate the moment ratios at 2nth order it is sufficient in the isotropic case to consider only the angular contribution  $\overline{\Omega} = \overline{\cos^{2p} \epsilon_1 \sin^{2q} \epsilon_1}$ . Performing the average over all solid angles we integrate  $d\Omega = -d(\cos \epsilon_1) d\epsilon_2$  with the uniform probability distribution  $P(\cos \epsilon_1, \epsilon_2) = 1/4\pi$ ,

$$\begin{aligned} \overline{\Omega} &= \int_0^\pi \int_0^{2\pi} P(\cos \epsilon_1, \epsilon_2) \cos^{2p} \epsilon_1 \sin^{2q} \epsilon_1 \sin \epsilon_1 d\epsilon_1 d\epsilon_2 \\ &= \frac{1}{4\pi} \int_0^{2\pi} \int_{-1}^1 \cos^{2p} \epsilon_1 \sin^{2q} \epsilon_1 d\cos \epsilon_1 d\epsilon_2 \\ &= \frac{1}{2} \int_{-1}^1 X^{2p} (1 - X^2)^q dX. \end{aligned} \quad (\text{A5})$$

The ratio of tangential moments at fixed order is thus,

$$\frac{m_{a,b}}{m_{c,d}} = \frac{a! \Gamma(a + \frac{1}{2})}{d! \Gamma(c + \frac{1}{2})} \quad (\text{A6})$$

where we have used the compact notation (39) and the fact that for moments of equivalent order  $a + b = c + d$ . The expression is simplified further for *adjacent* ( $a = c - 1$ ,  $b = d + 1$ ) moment ratios wherein the shift property of gamma functions may be exploited,

$$\frac{m_{p,q}}{m_{p+1,q-1}} = \frac{2q}{2p+1}. \quad (\text{A7})$$

Radial moment ratios may then be expressed as the product of adjacent moment ratios,

$$\frac{m_{n-q,q}}{m_{n,0}} = \prod_{i=0}^{q-1} \frac{m_{n-q+i,q-i}}{m_{n-q+i+1,q-i-1}} \quad (\text{A8})$$

$$= \prod_{k=1}^q \frac{k}{(n - q - \frac{1}{2} + k)} = \frac{q!}{(n - q + \frac{1}{2})_q} \quad (\text{A9})$$

and we note crucially that the order dependence of the isotropic system is the same as that of the separable augmented density system. This factor will be used to generalise the factor of  $1/2$  present in the second order Binney anisotropy parameter.

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